On Numerical Simulation of Linear Acoustic and Electromagnetic Waves in Unbounded Domains

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Abstract. We study the problem of constructing local absorbing boundary conditions (ABCs) for the numerical simulation of n-dimensional linear acoustic and electromagnetic waves in unbounded domains. Employing the technique of dimensional splitting, we reduce the construction of the ABCs for the original n-D equation to seeking for an ABC for the one-dimensional wave problem. Then, applying the Laplace transform in time as well as using spline interpolation for the initial data, we obtain an infinite family of functions that approximate the far-field solution with higher and higher accuracy. These functions are used as ABCs. Due to the property of compactness of supports of the splines, the boundary conditions appear to be local both in time and in space, which is extremely important for their numerical implementation. Yet, these ABCs are uncritical to the shape of the artificial boundary and therefore usable for simulating a vast number of practical wave problems in domains of drastically complex geometries. The resulting boundary value problems are well-posed in the sense of existence, uniqueness and stability of solution. Results of the numerical experiments confirm the theoretical study.

1 Introduction

Numerical solution to differential problems in unbounded domains requires introducing a finite computational domain enclosed by an artificial boundary. Indeed, all the matrices and vectors employed to compute the solution must be of finite sizes, and therefore the necessity in truncation of the original domain is dictated by the evident limitations on the computer memory. Hence, it is required to impose adequate boundary conditions for simulating the solution at the boundary points. The adequacy of the boundary conditions implies that 1) the resulting boundary value problem (BVP) must be well-posed in the sense of existence, uniqueness, and stability of solution, and 2) the error between the solutions to the original Cauchy problem and the resulting BVP should be as small as possible in the domain of interest. Boundary conditions satisfying the aforesaid requirements are often called artificial boundary conditions (ABCs) [17].

A special class of ABCs is known as one-way, or absorbing, or non-reflecting boundary conditions (NRBCs) [3, 16]. These boundary conditions appear when studying various wave phenomena in such fields of scientific computing as geophysics [1, 4, 8], theory of elasticity [5, 10, 13], computational fluid dynamics [3, 4, 15], and others. NRBCs are derived in such a manner that permit waves propagating outward the computational domain only, while no propagation towards the interior is allowed.

For the last three decades there have been developed a whole series of different methods for the construction of NRBCs [2, 4, 5, 8, 10, 11, 13–15] (see also [17]). However, all of them lead to boundary conditions that suffer from some or other disadvantages. For example, many methods provide BCs that can be used with a planar (i.e., rectangular) artificial boundary only; others lead to NRBCs that are non-local either in time or in space, or even both, and hence these BCs are unrealisable from the computational standpoint; third group of methods is oriented to solving a particular class of equations, and therefore fails when applying to problems of other types.

In this paper we present an advanced methodology for the construction of non-reflecting boundary conditions. The key idea consists in the employment of the techniques of operator factorisation and dimensional splitting [12]. Unlike all other NRBCs, ours are geometrically universal and besides appear local both in time and in space. These properties allow to use the approach when numerically solving a wide spectrum of practical wave problems in domains of extremely complex geometries [7]. In addition, the constructed boundary conditions are more accurate than many others, e.g., those derived in [4].

2 Problem Formulation

Consider the n-dimensional wave equation

$$Au \equiv \frac{\partial^2 u}{\partial t^2} - a^2 \Delta u = f, \quad (\mathbf{x},t) \in \mathbf{R}^n \times (0,+\infty) \tag{1}$$

subject to the initial condition

$$u\big|_{t=0} = g(\mathbf{x}), \quad \frac{\partial u}{\partial t}\bigg|_{t=0} = 0 \ . \tag{2}$$

Here $u = u(\mathbf{x},t)$ is the function to be sought, $a = a(\mathbf{x},t) \geq 0$ is the wave velocity, Δ is the Laplacian in $\mathbf{x}$, and $f = f(\mathbf{x},t)$ denotes the sources. Let the computational domain be an open region $\Omega \subset \mathbf{R}^n$ with a piecewise smooth boundary Γ. We assume $a = \text{const}$ outside Ω, as well as $\operatorname{supp} f \subseteq \Omega$. In order to solve problem (1)-(2) numerically, it is required to construct a differential operator B for the boundary condition $Bu\big|_{\mathbf{x}\in\Gamma} = 0$; upon this, the operator B must be derived in such a manner that waves propagating from Ω leave the domain without reflections to the inside, that is the boundary condition must be *non-reflecting*.

3 Construction of the ABCs

We shall construct the boundary operator B in two steps. At the first step we employ the method of dimensional splitting [12]: we choose some i from the range $[1,\dots,n]$, freeze the coordinates $\{x_j\}_{j=1,j\neq i}^{n}$, and consider equation (1) on Γ with respect to x_i only, i.e.

$$A_i u \equiv \frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x_i^2} = 0, \quad (x,t) \in \Gamma \times (0,+\infty), \quad \{x_j\}_{j=1,j\neq i}^{n} \text{ are fixed}. \tag{3}$$

At the second step, in order to satisfy the requirement of no reflection of the boundary condition, we factorise the operator A_i onto the incoming and outgoing components— $A_i = A_i^+ A_i^-$ —and rewrite equation (3) the form

$$A_i^+ A_i^- u = \left(\frac{\partial}{\partial t} + a\frac{\partial}{\partial x_i}\right)\left(\frac{\partial}{\partial t} - a\frac{\partial}{\partial x_i}\right) u = 0 \ . \tag{4}$$

It is important to emphasise that we decoupled the incoming and outgoing waves in the *original, physical* space $\mathbf{R}^n$, without involving any additional techniques like Fourier transform [4] or others. Therefore, we avoided imposing undesired restrictions on the shape of the boundary Γ, and hence, on the applicability of the subsequent NRBCs too.

The derivation of the boundary operator is now obvious: if the incoming wave at the point x_i (all the other x_j's are fixed) is represented by the positive component A_i^+, then we prohibit it defining $A_i^+ = E$ (here E is the identity operator) and obtain

$$B_i u \overset{x\in\Gamma}{=} A_i^+ A_i^- u = EA_i^- u = \left(\frac{\partial}{\partial t} - a\frac{\partial}{\partial x_i}\right) u = 0 \ ; \tag{5}$$

otherwise, if the incoming wave corresponds to A_i^-, it holds $B_i = \partial_t + a\partial_{x_i}$.

In fact, the construction of the one-dimensional NRBC is complete, and now we can derive the general n-D absorbing boundary condition. However, before doing this let us construct a family of 1D non-reflecting boundary conditions basing on the original BC (5). This family will be derived in such a way that its members will be approximate solutions to equation (5) specially adapted to finite difference implementation. So, involving the technique used in [6], we apply to (5) the Laplace transform in time and then approximate the initial condition g from (2) by a family of splines. Next, assuming the image of u to be bounded in x_i in the dual space, we apply the inverse Laplace transform and thus come to an infinite family of approximate solutions to equation (5) in the physical space [7]:

$$u^{[k+1]} = u^{[k]} + \sum_{p=1}^{m} a^p \frac{\tau_k^p}{p!} \frac{\partial^p u^{[k]}}{\partial x_i^p}, \quad \tau_k = t_{k+1} - t_k, \quad u^{[0]} = g, \quad k \in \{0\} \cup \mathbb{N} \,. \tag{6}$$

Here $m \geq 1$ is the spline order. Consequently, the general n-dimensional absorbing boundary condition has the form

$$u^{[k+1]}(\mathbf{x}) = u^{[k]}(\mathbf{x}) + \sum_{p=1}^{m} a^p \frac{\tau_k^p}{p!} \sum_{i=1}^{n} s_i^p \frac{\partial^p u^{[k]}(\mathbf{x})}{\partial x_i^p}, \quad \mathbf{x} \in \Gamma^{[0]} \,, \tag{7}$$

where $s_i = s_i(\mathbf{x})$ is the sign function having the value +1 if the incoming wave at the point $\mathbf{x} \in \Gamma$ in the direction x_i is given by the operator A_i^-, and −1 otherwise (see formula (5)). For the errors produced by solutions (7) on the boundary we have the estimates

$$|\varepsilon_m| \leq a \left(\xi_1 \frac{(a\tau_k)^m}{m!} \sum_{i=1}^{n} s_i^{m+1} + \xi_2 \sum_{p=1}^{m} \frac{(a\tau_k)^p}{p!} \sum_{j=1}^{n} \sum_{i=1, i \neq j}^{n} s_j s_i^p \right), \tag{8}$$

where $\xi_1 = \max_i \left| \frac{\partial^{m+1} u^{[k]}(\mathbf{x})}{\partial x_i^{m+1}} \right|$ and $\xi_2 = \max_{p,i,j} \left| \frac{\partial^{p+1} u^{[k]}(\mathbf{x})}{\partial x_j \partial x_i^p} \right|$. The following proposition holds.

Proposition 1. *For every $m \geq 1$ the corresponding wave boundary value problem is well-posed in the sense of existence, uniqueness, and stability of solution.*

We shall omit a formal proof of this statement, and only mention that it can be done by means of functional analysis and the theory of generalised functions. For further studies on well-posedness of the resulting BVPs the interested reader may refer to [7].

Let us make two important remarks. First of them concerns the use of splines when approximating the initial data g in the Laplace-transformed equation (5). Specifically, it is essential that we employed *compactly* supported basis functions rather than some *infinitely* supported ones [15]. Due to this we again avoided restrictions on the shape of the boundary Γ, and so derived a family of *geometrically universal* NRBCs; in addition, these BCs are local and adapted to numerical implementation in time.

Another remark relates to the factorisation of the operator A_i in (4) and the subsequent prohibition of the incoming wave at the boundary point. Namely, it can easily be observed that having defined $A_i^- = E$ (or $A_i^+ = E$) we, properly speaking, disregarded the *second* order of the temporal derivative of the wave equation and thus reduced it from two to one. Therefore, the resulting boundary conditions (7) serve only in case of *zero* initial condition $\partial_t u|_{t=0} = h(\mathbf{x})$, as defined in (2). However, to generalise these NRBCs one may perform a preliminary change of variable (with respect to the function u) and to transfer $h \neq 0$ to the right-hand side of equation (1). This will allow to subsequently take into account the presence of the non-zero initial data and hence to construct a more general version of the operator B [7].

4 Numerical Results

We tested the developed method having performed several experiments on functionality and efficiency of the constructed NRBCs. For clearness of presentation of the results we considered the case $n = 2$.

First of all we compared our boundary conditions with those derived in the classical paper by Engquist and Majda [4]. For this we solved the original Cauchy problem (1)-(2) and a few boundary value problems with various boundary conditions. Specifically, we tested the lowest-order BC (7) corresponding to $m = 1$ and the first two NRBCs from [4] analysed by the authors in detail. The artificial boundary was supposed to be the straight line $\Gamma = \{x_2 = 0\}$, the domain of interest was $\Omega = (-6,6)\times(0,6)$, and we chose $g(x) = \sin^2\pi(x_1 + 0.5)\sin^2\pi(x_2 - 0.5)$, $\operatorname{supp} g \subset [-0.5,0.5]\times[0.5,1.5]$, $a = 1$, $\Delta x_1 = \Delta x_2 = 5\tau = 0.05$. To discretise the equation in Ω we used the standard second-order "wave" finite difference scheme [9, p. 228], while on the lateral boundary we imposed the zero Neumann boundary condition. At each time moment on Γ we computed maximum of the reflected wave related to maximum of the incident wave.

In Fig. 1 there are two wave profiles corresponding to the solutions to the infinite-domain problem and the BVP with BC (7). It can be seen that the second profile is similar to the first one, with a small reflection as well. Expressed numerically, Table 1 summarises L_2-norms of the relative error for different angles of wave incidence. One may observe that there are considerable benefits in the precision of solution, especially for large values of θ, under substantial geometrical flexibility of our NRBCs.

Table 1. Planar artificial boundary: relative error (in %) for different angles of wave incidence

Angle (θ)	BC (7)	1st E-M	2nd E-M
0°	2.55	6.06	2.10
10°	2.50	6.07	2.18
20°	2.56	6.33	2.40
30°	3.55	7.91	3.14
40°	5.65	10.76	4.22
50°	6.50	15.40	6.11
60°	7.28	21.99	9.54
70°	7.53	30.36	16.14

To try the method on complex geometries, we considered the domain Ω shown in Fig. 2 (top) and repeated the experiments computing the relative error between amplitudes of the reflected and incident waves. As in the case of planar artificial boundary, there was a weak reflection from Γ, and the error did not exceed 2.5% for $\theta = 0^\circ$ and 7.9% for $\theta = 70^\circ$. At the bottom of Fig. 2 we show the numerical solution at $T = 0.33$ as well.

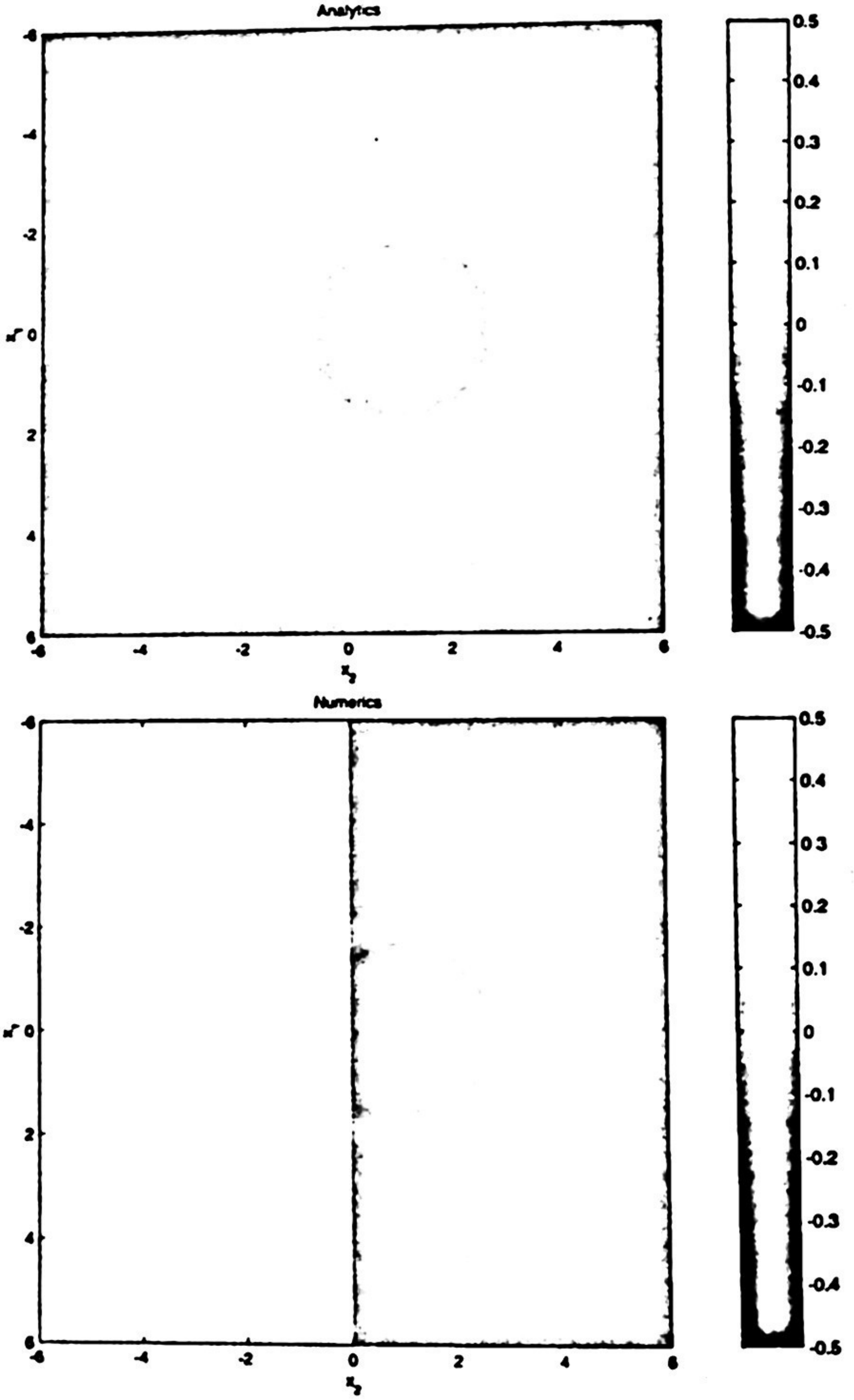

Fig. 1. Planar artificial boundary: solutions to the original Cauchy problem (top) and the resulting BVP (bottom) at $T = 2.0$

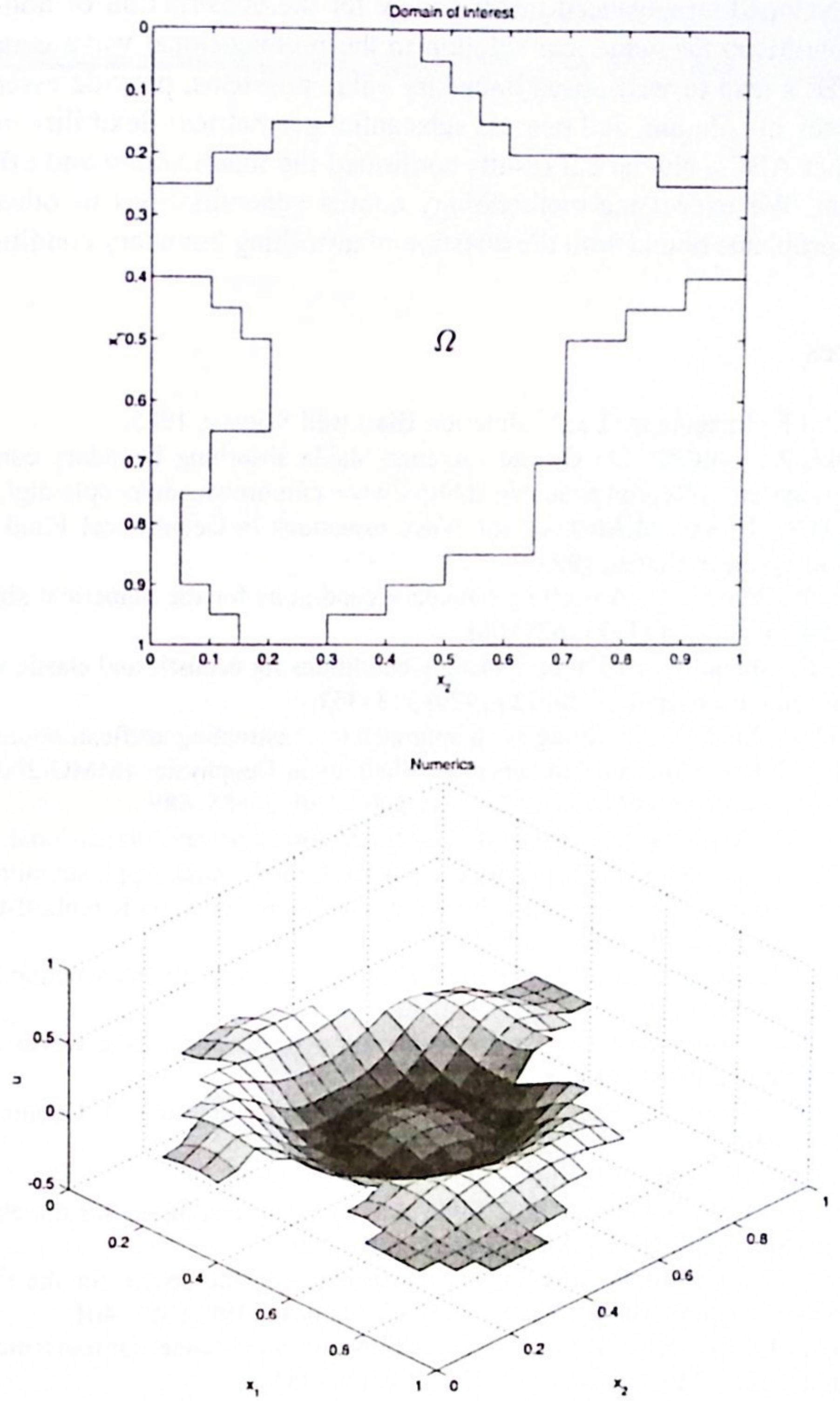

Fig. 2. Experiments with complex geometries: the domain of interest (top) and the BVP solution at $T = 0.33$ (bottom)

5 Conclusions

We have developed an advanced methodology for the construction of non-reflecting boundary conditions for numerical solution to the n-dimensional wave equation. The derived NRBCs lead to well-posed boundary value problems, provide essential gains in the accuracy of solution, and possess substantial geometrical flexibility in comparison with other ABCs. Numerical results confirmed the functionality and efficiency of the approach. We expect the methodology admits generalisations to other types of differential problems bound with the question of absorbing boundary conditions.

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